Liberian Math Olympics Solutions

July 4, 2013

1 Individual Round

1. A store sells the jerseys of Barca, Real Madrid, Manchester United, and Chelsea. Each jersey is sold in three sizes: small, medium, and large. How many different types of jerseys are there?

   Solution: There are a total of 4 teams: Barca, Real Madrid, Manchester United, and Chelsea. Each team's jersey can appear in 3 sizes: small, medium and large. Because each of the 4 teams has 3 sizes, there are a total of $4 \times 3 = \boxed{12}$ types of jerseys.

2. What is $\sqrt{3^2 \times 4^2}$?

   Solution: $\sqrt{3^2 \times 4^2} = \sqrt{3^2} \times \sqrt{4^2} = 3 \times 4 = \boxed{12}$

3. In 2014, the 4th of July will be a Friday. What day of the week will the 4th of July be in 2015?

   Solution: July 4, 2015 is 365 days after July 4, 2014. 365 days is 52 weeks and 1 day, so July 4, 2015 falls one day later in the week than July 4, 2014, namely, on a $\boxed{\text{Saturday}}$.

4. A prime number is a positive integer that has exactly two different divisors, 1 and itself. What is the smallest prime number?

   Solution: 1 only has one divisor; 2 is divisible by exactly 1 and 2, so the answer is $\boxed{2}$.

5. An umbrella costs 225 LD. If the exchange rate is 1 USD for 75 LD, then how many USD does the umbrella cost?

   Solution: Since 1 USD exchanges for 75 LD, one can divide 225 LD by $75\frac{\text{LD}}{\text{USD}}$ to get $\boxed{3 \text{ USD}}$. Another way to solve the problem is to see that $75 \times 3 = 225$, so one would need 3 USD to get 225 LD.

6. What is the probability that a random integer in $\{1, 2, 3, \ldots, 10\}$ is a multiple of 3?

   Solution: The multiples of 3 in $\{1, 2, 3, \ldots, 10\}$ are $\{3, 6, 9\}$, so there are a total of 3 multiples of 3. Since there are 10 numbers in the set $\{1, 2, 3, \ldots, 10\}$, the probability of choosing a multiple of 3 is $3/10$, or $\boxed{30\%}$.

7. I am thinking of a two-digit number. The sum of the digits is 11. One digit is 5 more than the other one. The number is odd. What number am I thinking of?

   Solution: Let the smaller digit be a. Then the larger digit is $a + 5$, and so $a + a + 5 = 2a + 5 = 11$. Solving gives $a = 3$, so the digits are 3 and 8, and the number is $\boxed{83}$.

8. Six people are in a room. If each person shakes hands with each other person exactly once, then how many total handshakes are there?

Solution: We start by numbering the people $1, 2, \ldots, 6$. We want to count the number of unordered pairs of distinct people. One way to do so is to write all the pairs out: $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}$. There are $\boxed{15}$ pairs. Alternatively, there are $6 \cdot 5 = 30$ *ordered* pairs of distinct people, but each unordered pair is counted twice (e.g. we're counting both $(5,6)$ and $(6,5)$ when we only want to count $\{5,6\}$), so we divide by 2 to get the correct answer.

9. Compute 90% of 50% of $\frac{1}{3}$ of 360.

Solution: $.9 \times .5 \times \frac{1}{3} \times 360 = \boxed{54}$.

10. In $\triangle ABC$, $\angle A = 40°$, $\angle B = 70°$, and $\angle C = 70°$. Additionally, $AB = 4$. What is the value of AC?

Solution: Because $\angle B = \angle C$, $\triangle ABC$ is an isosceles triangle where the two sides opposite to the two congruent angles are congruent. AB is opposite to $\angle C$ and AC is opposite to $\angle B$, so $AB = AC = \boxed{4}$.

11. On his first 4 math exams, Mohammed scores 90, 95, 96 and 88. On his fifth exam, he does not study and does poorly. His average on the 5 exams is 84. What did he score on his fifth exam?

Solution: Let Mohammed's score on his last exam be x. Then $\frac{90+95+96+88+x}{5} = 84$. Solving gives $x = 61$, so he scored a $\boxed{61}$ on his last exam.

12. What is the length of the longest straight line that can be drawn within a square that has sides of length 2?

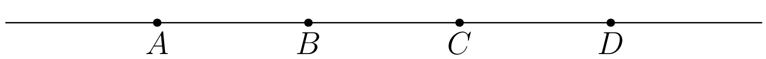
Solution: The longest line is the diagonal, drawn here. By the Pythagorean theorem, it has length $\sqrt{2^2 + 2^2} = \sqrt{8} = \boxed{2\sqrt{2}}$.

13. What is the product of all of the integers from -7 to 7, inclusive? That is, compute

$$(-7)(-6)(-5) \cdots 5 \cdot 6 \cdot 7.$$

Solution: The product includes a factor of 0, so the answer is $\boxed{0}$.

14. A, B, C, D are points in order on a line. $AC = 8$, $BD = 7$, and $BC = 3$. What is AD?



(not to scale)

Solution: $AB = AC - BC = 5$. $CD = BD - BC = 4$. $AD = AB + BC + CD = 5 + 3 + 4 = \boxed{12}$.

15. A car wheel is 3 feet in diameter. How many rotations will the wheel make as the car drives 1 mile? (There are 5280 feet in one mile.) Write your answer in terms of π.

Solution: The circumference of the wheel is 3π, so the total number of rotations is $\frac{5280}{3\pi} = \boxed{1760/\pi}$.

16. A circle of diameter 1 is drawn within a square of side length 1. To the nearest percent, what percent of the square lies outside the circle? (You may use the fact that $\pi \approx 3.14$.)

Solution: The area of the circle is $\pi/4 \approx 0.785$, so the area that lies outside the circle is $1 - 0.785 = 0.215$. Therefore, about 21.5% of the square lies outside the circle. Because π is slightly greater than 3.14, the nearest percent is $\boxed{21\%}$, but we accepted 22% as well.

17. An equilateral triangle that has three sides of length 1 has area $\frac{\sqrt{3}}{4}$. What is the area of an equilateral triangle with three sides of length 4?

Solution: One can divide the triangle with sides of length 4 into four congruent equilateral triangles with sides of length 2. One can do the same again and divide each of the four smaller equilateral triangles into even smaller equilateral triangles with sides of length 1. So, we get a total of $4 \times 4 = 16$ equilateral triangles with sides of length 1. Therefore, the sum of the areas of the 16 triangles with sides of length 1 must equal the area of the triangle with sides of length 4, so the area of the original triangle has an area of $16 \times \frac{\sqrt{3}}{4} = \boxed{4\sqrt{3}}$.

18. A palindrome is a number that is the same read backwards as read forwards. 3993, 6008006, and 12321 are examples of palindromes. 330 is not a palindrome: when read backwards, it is $033 = 33$. How many palindromes are 4 digits long and greater than 1000?

Solution: A 4-digit palindrome is of the form $abba$, where a and b are digits and $a \neq 0$. There are 9 different choices for a and 10 different choices for b, so there are $9 \times 10 = \boxed{90}$ different ways to construct a palindrome by choosing a and b.

19. A rhombus is a quadrilateral with all sides of equal lengths. Rhombus $ABCD$ has two diagonals. Diagonal AC has length 12 and diagonal BD has length 20. What is the area of the rhombus?

Solution: The diagonal of length 20 divides the rhombus into two triangles, each with base 20 and height $12/2 = 6$. So, the area of the rhombus is $2 \cdot \left(\frac{1}{2}20 \cdot 6\right) = \boxed{120}$.

20. If all men work at the same rate and 4 men can build 4 houses in 20 days, how long will it take for 27 men to build 15 houses?

Solution: 1 man can build 1/20 of a house per day. If 27 men build 15 houses, then 1 man needs to build 9/5 houses, which will take $\frac{9}{5} \div \frac{1}{20} = \boxed{36}$ days.

2 Team Round

1. Compute

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6}.$$

Solution: One can cancel out the common terms in the numerator and denominator. So, 2 in the numerator cancels out the 2 in the denominator, 3 in the numerator cancels out the 3 in the denominator, and so on with 4 and 5, until the expression that is left is $\boxed{\frac{1}{6}}$.

2. What is the surface area of a cube with edges of length $\frac{1}{2}$?

Solution: The surface area is $6 \cdot (1/2)^2 = \boxed{3/2}$.

3. $5x - 2y = 26$ and $3x + 10y = -18$. Find $x + y$.

Solution: Adding the two equations gives $8(x + y) = 8x + 8y = 8$, so $x + y = \boxed{1}$.

4. The distance between Kakata and Monrovia is 45 miles. Vofee is walking from Kakata to Monrovia at 2 miles per hour. Abe is walking on the same road from Monrovia to Kakata at 3 miles per hour. They both start walking at $5:00$ AM and do not rest until they meet. At what time will they meet each other on the road?

Solution: Their relative speed is 5 miles per hour, so they meet in 9 hours, at which point it is $\boxed{2:00}$ PM.

5. If I add up the angle measures of a regular octagon, how many degrees do I get?

Solution: One can draw a line between the opposite angles in the octagon. These 4 lines will divide the octagon into 8 triangles. Each triangle has 3 angles, but only 2 of the 3 coincide with the angles of the octagon; the remaining angle is at the center of the triangle and 8 of these angles form a circle. So, we can add all the angles in the triangles and then subtract 360 degrees to get the the sum of the angles of the octagon. $8 \times 180 - 360 = \boxed{1080 \text{ degrees}}$.

6. If a book has 100 pages, and every page has a page number, how many times will the digit 7 appear in the page numbers of the book?

Solution: It will appear in the ones place in $7, 17, \ldots, 97$ (10 times) and in the tens place in $70, 71, \ldots, 79$ (10 times), for a total of $\boxed{20}$ 7s.

7. Consider the number

$$12345678910112131415 \cdots 979899100,$$

formed by writing the first 100 positive integers in order. What is its 100th digit?

Solution: $1 - 9$ are the first 9 digits. Afterwards, we have the two-digit numbers $10 - 54$. There are 45 of these two-digit numbers, for a total of 90 digits. So all in all, from 1 to 54, we have 99 digits. Thus the 100th digit is $\boxed{5}$, the first digit of 55.

8. Compute $1 + 2 + 3 + \cdots + 98 + 99 + 100$.

Solution: Write the sequence backwards as $100 + 99 + 98 + \cdots + 3 + 2 + 1$. The sum of these two sequences is $101 + 101 + 101 + \cdots + 101 + 101 + 101 = 101 \cdot 100 = 10100$, so the sum of the sequence is $10100/2 = \boxed{5050}$

9. The last digit of 3^3 is 7, since $3^3 = 27$. What is the last digit of 3^{100}?

Solution: The last digit of 3^n is equal to the last digit of 3 times the last digit of 3^{n-1}. Therefore, using "$\equiv$" to denote "has last digit":

$$3^0 \equiv 1$$

$$3^1 \equiv 3$$

$$3^2 \equiv 9$$
$$3^3 \equiv 7$$
$$3^4 \equiv 1$$
$$3^5 \equiv 3$$
$$3^6 \equiv 9$$
$$3^7 \equiv 7.$$

We see that the last digit is periodic with period 4. Therefore, since 100 is a multiple of 4, the last digit of 3^{100} is equal to the last digit of 3^0, which is $\boxed{1}$

10. How many numbers in $\{1, \cdots, 200\}$ are divisible by 3 or 5?

Solution: The numbers in the set that are divisible by 3 are $\{3, 6, \ldots, 198\}$. There are $198/3 = 66$ of them. The numbers in the set that are divisible by 5 are $\{5, 10, \ldots, 200\}$, and there are $200/5 = 40$ of them. If we just add $66 + 40 = 106$, our answer is too large: we counted all of the numbers divisible by both 3 and 5 twice, namely $\{15, 30, \ldots, 195\}$, and there are $195/15 = 13$ of those. So, our answer is $66 + 40 - 13 = \boxed{93}$.